Packing Equal Circles in a Square — bounds, minimal polynomials and classification

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In sciences, engineering and real life several problems lead to the question of finding the densest packing of equal objects in a bounded region of a special geometrical shape. Sometimes these kind of questions can be generalized in the following way: Which is the largest $\overline{m}_n$ distance of n distinct points, so that all points are in a compact convex subset of the Euclidean plane and the distance between any two of them greater or equal than $\overline{m}_n$. If one considers the points as centers of n circles with equal radii, the problem is equivalent to determine the largest radius $\overline{r}_n$ these circles can have, neither overlapping each other nor putting off the region.

This work studied the following problem: Locate n equal and non-overlapping circles in a square, such that the radius of the circles be maximal. Originally this question arise from the discrete geometry, but it is in connection with the subject of facility location theory in operations research too. In the investigation many branches of mathematics and operations research meet: deterministic and stochastic optimization, numerical mathematics, interval mathematics, graph theory, Groebner bases theory, number theory, etc.

This investigation organized for three subjects: improving the theoretical bounds, algebraic investigation of minimal polynomials of packings, and studying a classification of circles packing based on minimal polynomials.

Up to $n = 5$ circles the problem is trivial and there are solutions for n=6, 8, 9, 14, 16, 25 and 36 using only mathematical tools. Since 1990 proof of optimality of circles packing were made by computer aided methods. Using deterministic optimization techniques, the optimal packings are known up to $n = 30$. There are theoretical lower and upper bounds of $\overline{m}_n$ and $\overline{r}_n$, I have improved some of them.

THEOREM 1. For all $n \geq 2$

$$\sqrt{\frac{2}{\sqrt{3}n}} < \overline{m}_n,$$

$$\overline{r}_n \leq \min\left(\frac{1}{\sqrt{2\sqrt{3}n + (4\lfloor\sqrt{n}\rfloor - 2)(2-\sqrt{3})}}, \frac{1+\sqrt{1+\frac{2}{\sqrt{3}}(n-1)}}{2n + 2\sqrt{1+\frac{2}{\sqrt{3}}(n-1)}}\right).$$

Stochastic optimization methods can be used to find approximate packings for higher n values. It is important to realize that an approximate packing found by the computer is not always sure its existence in mathematical sense. The structure suggested by the numerical result is only a kind of conjecture, because the rounding errors can produce serious mistakes. We have to prove that the structure of a given packing really exists. A possible approach for the proof is to find the corresponding suitable quadratic system of equations to the packing and try to solve it. Sometimes the computer algebra systems can help the investigation based on algebraic, symbolic computations.

An interesting parameter of circles packing is its minimal polynomial. The minimal polynomial $P_n(m)$ of a packing is a polynomial with minimal degree and integer coefficients, where the first positive root of the polynomial is $\overline{m}_n$. I have given more possible way to determine a minimal polynomial of a packing.

Based on minimal polynomials can be give an exact classification of optimal packings accord to the structure of packings.

References

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